数理解析研究所講究録 1390 巻 2004 年 1-8

On Generic Predicates and Automorphisms

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Abstract

We prove that the class of the generic automorphisms of unstable structures constructed from stable structures by adding a generic predicate is not elementary. We also give Some discussion on generic automorphisms of a generic automorphism.

Introduction

Given a complete, model complete theory T in a language $\mathcal{L}$, we consider the theory $T_\sigma = T \cup \{$"σ is an $\mathcal{L}$-automorphism"$\}$ in the language $\mathcal{L} \cup \{\sigma\}$. For M a model of T, and $\sigma \in \mathrm{Aut}_{\mathcal{L}}(M)$ we call σ a generic automorphism of M if (M, σ) is an existentially closed model of T_σ.

It is known that the class of generic automorphism of T is not elementary if T is unstable with the PAPA [4], has the strict order property [5], or T does not eliminate the $\exists^\infty$ [4]. We conjecture that this class is not elementary if T is unstable. But we don't even know how to handle the general simple unstable case. We will consider simple unstable theories constructed from stable theories by adding a generic predicate or a generic automorphism. We try to show that the class of the generic automorphisms of the models of a theory constructed this way is not elementary. We have succeeded to show it in case of generic predicates but not in case of generic automorphisms. Nevertheless, we will give some arguments concerning two commuting automorphisms.

1 Preliminaries

In this paper, we work in a big model of some theory. a, b, etc. denote tuples of elements of the universe, A, B, etc. denote a small subset of the universe, and x, y, etc. denote tuples of variables. If a is a tuple and A is a set, $a \in A$ means that each entry of a belongs to A. We don't usually distinguish by notation between a tuple a and the set of the entries of a.

Suppose $\mathcal{L}$ is a language. $\mathrm{acl}_{\mathcal{L}}(A)$ denote the set of the elements satisfying some algebraic formula in $\mathcal{L}$ with parameters in A. We write $\mathrm{acl}(A)$ for $\mathrm{acl}_{\mathcal{L}}(A)$ if $\mathcal{L}$ is clear from the context. $\mathrm{dcl}_{\mathcal{L}}(A)$ denote the set of the elements satisfying some algebraic formula in $\mathcal{L}$ with parameters in A with only one solution.

If $\mathcal{L}$ is a language and σ, τ, P are new non-logical symbols, $\mathcal{L}_P = \mathcal{L} \cup \{P\}$, $\mathcal{L}_\sigma = \mathcal{L} \cup \{\sigma\}$, $\mathcal{L}_{P,\sigma} = \mathcal{L} \cup \{P, \sigma\}$, and $\mathcal{L}_{\sigma,\tau} = \mathcal{L} \cup \{\sigma, \tau\}$.

We list some known facts needed later.

Definition 1.1 Let T be a theory in a language $\mathcal{L}$. We say that T has the *PAPA* (la propriété d'amalgamation pour les automorphismes) if $M_0, M_1, M_2 \models T$, $\sigma_1 \in \mathrm{Aut}_{\mathcal{L}}(M_1)$, $\sigma_2 \in \mathrm{Aut}_{\mathcal{L}}(M_2)$, and $\sigma_1|M_0 = \sigma_2|M_0$ then there are $M_3 \models T$, $\sigma_3 \in \mathrm{Aut}_{\mathcal{L}}(M_3)$, and $h : M_2 \to M_3$ such that $h|M_0$ is the identity on M_0, $\sigma_3|M_1 = \sigma_1$ and $\sigma_3|h(M_2) = h\sigma_2 h^{-1}$.

Fact 1.2 ([4]) *Let T be a complete theory in a language $\mathcal{L}$. If T is model complete, unstable and has the PAPA, then T_σ has no model companion in $\mathcal{L}_\sigma$.*

Fact 1.3 (Chatzidakis, Pillay [2]) *Let T be a complete theory in a language $\mathcal{L}$ and suppose T is model complete. Then the model companion T_P^* of T in the language $\mathcal{L}_P$ exists if and only if T eliminates the quantifier $\exists^\infty$. If T_P^* exists then $(M, P) \models T_P^*$ if and only if (i) $M \models T$ and (ii) for every $\mathcal{L}$-formula $\varphi(x, z)$ where x is a n-tuple of variables, for every subset I of $\{1, \dots, n\}$, for any tuple $b \in M$, if there is $a = (a_1, \dots, a_n) \in M$ such that $a \cap \mathrm{acl}_{\mathcal{L}}(b) = \emptyset$ and $a_i \neq a_j$ for $i \neq j$, then there is $a' = (a'_1, \dots, a'_n) \in M$ such that $\varphi(a', b)$, $P(a'_i)$ for $i \in I$, and $\neg P(a'_i)$ for $i \notin I$.*

2 Theories with a Predicate and an Automorphism

The following lemma is a well-known fact.

Lemma 2.1 *Let T be a complete theory. Let a be a tuple and A a set such that $a \cap \mathrm{acl}(A) = \emptyset$ then for any $B \supset A$ there is a tuple $a' \models \mathrm{tp}(a/A)$ such that $a' \cap \mathrm{acl}(B) = \emptyset$.*

Proof. We prove this by induction on the length of a tuple a. We can assume that $A = \mathrm{acl}(A)$. If a is a single element, the conclusion follows by compactness.

Let $a = (a_1, a_2)$ where a_1 is a single element and a_2 a tuple. Suppose $\varphi(x, y) \in \mathrm{tp}(a_1, b/A)$ where x is a single variable, and $b_1, \dots, b_n$ are elements in $\mathrm{acl}(B) \setminus \mathrm{acl}(A)$. We show that there are a'_1, a'_2 such that $\varphi(a'_1, a'_2)$ and $(a'_1, a'_2) \cap \{b_1, \dots, b_n\} = \emptyset$. Then the conclusion follows by compactness.

We can choose $a'_2 \models \mathrm{tp}(a_2/A)$ such that $a'_2 \cap \{b_1, \dots, b_n\} = \emptyset$ by induction hypothesis. If there is $a'_1 \notin \{b_1, \dots, b_n\}$ such that $\varphi(a'_1, a'_2)$, we are done.

By way of contradiction, suppose for any c and d, $\varphi(c, d)$ and $d \cap \{b_1, \dots, b_n\} = \emptyset$ implies $c \in \{b_1, \dots, b_n\}$. Consider a formula $\psi(x)$ over A expressing that there are pairwise disjoint tuples $d_1, \ldots, d_{n+1}$ such that $\varphi(x, d_j)$ for $j = 1, \ldots, n+1$. We show that $\psi(x)$ is algebraic. Let b satisfy $\psi(x)$. Then there are pairwise disjoint tuples $d_1, \ldots, d_{n+1}$ such that $\varphi(b, d_j)$ for $j = 1, \ldots, n+1$. Since they are disjoint, some d_j is disjoint from $\{b_1, \ldots, b_n\}$. Therefore, $b \in \{b_1, \ldots, b_n\}$ by the hypothesis.

Hence, b_i satisfying $\psi(x)$ belongs to $\mathrm{acl}(A) = A$, and for b_i satisfying $\neg\psi(x)$, the number of pairwise disjoint solutions of $\varphi(b_i, y)$ is bounded by n.

By an iterated use of induction hypothesis, there are tuples $d_1, \ldots, d_{n^2+1}$ such that $d_j \models \mathrm{tp}(a_2/A)$ and $d_j \cap \mathrm{acl}(Ab_1, \ldots, b_n d_0 \ldots d_{j-1}) = \emptyset$ for $j = 1, \ldots, n^2+1$. In particular, the d_j's are disjoint each other. For each b_i satisfying $\neg\psi(x)$, at most n tuples among the d_j's satisfy $\varphi(b_i, y)$. Therefore, for some d_j, $\neg\varphi(b_i, d_j)$ holds for any b_i satisfying $\neg\psi(x)$. Let $d = d_j$. Since d and a_2 are conjugate over A, there is an element c such that (c, d) and (a_1, a_2) are conjugate over A. Therefore, $\varphi(c, d)$ and $c \notin A$. Hence, $c \neq b_i$ for any b_i. This is a contradiction. □

Theorem 2.2 *Let T be a complete theory in a language $\mathcal{L}$. Suppose T is model complete and the model companion T_P^* of T in the language L_P exists. Then any model of $T_{P,\sigma} = T \cup \{\sigma$ is an L_P-automorphism$\}$ embeds in a model of $(T_P^*)_\sigma = T_P^* \cup \{\sigma$ is an L_P-automorphism$\}$. In particular, they have the same class of the existentially closed models. Therefore, $T_{P,\sigma}$ has a model companion if and only if $(T_P^*)_\sigma$ has one, and they are the same if they exist.*

Proof. We work in a big model $\mathcal{M}$ ($\mathcal{L}$-structure) of T.

Claim 2.2.1 *Suppose (M, σ_M) is a model of T_σ and a, b are tuples in M such that $a \cap \mathrm{acl}_{\mathcal{L}}(b) = \emptyset$. Then there is a sequence $\langle a_i : 0 \leq i < \omega\rangle$ such that $\sigma(\mathrm{tp}_{\mathcal{L}}(\langle a_i : 0 \leq i < \omega\rangle/M)) = \mathrm{tp}_{\mathcal{L}}(\langle a_i : 1 \leq i < \omega\rangle/M)$, $a_i \cap \mathrm{acl}_{\mathcal{L}}(Ma_0 \ldots a_{i-1}) = \emptyset$ for each i, and $a_0 \models \mathrm{tp}(a/b)$.*

We construct such a sequence by induction. By Lemma 2.1, there is $a_0 \models \mathrm{tp}_{\mathcal{L}}(a/b)$ such that $a_0 \cap M = \emptyset$. Again by Lemma 2.1, there is $a_1 \models \sigma_M(\mathrm{tp}_{\mathcal{L}}(a_0/M))$ such that $a_1 \cap Ma_0 = \emptyset$.

Suppose we have constructed a sequence $\langle a_i : 0 \leq i < n\rangle$ such that

$$\begin{aligned} \sigma_M(\mathrm{tp}_{\mathcal{L}}(a_0, \ldots, a_{n-2}/M)) &= \mathrm{tp}_{\mathcal{L}}(a_1, \ldots, a_{n-1}/M) \text{ and} \\ a_i \cap \mathrm{acl}_{\mathcal{L}}(Ma_0 \ldots a_{i-1}) &= \emptyset \end{aligned}$$

for $i < n$. let $\sigma' \in \mathrm{Aut}_{\mathcal{L}}(\mathcal{M})$ be a extension of σ_M such that $\sigma'(a_0, \ldots, a_{n-2}) = (a_1, \ldots, a_{n-1})$. By Lemma 2.1, we can choose $a_n \models \sigma'(\mathrm{tp}_{\mathcal{L}}(a_{n-1}/Ma_0 \ldots a_{n-2}))$ such that $a_n \cap \mathrm{acl}(Ma_0 \ldots a_{n-1}) = \emptyset$. Therefore, there is an $\mathcal{L}$-automorphism σ_n of $\mathcal{M}$ such that σ_n extends σ' and $\sigma_n(a_{n-1}) = a_n$. We have Claim 2.2.1.

Claim 2.2.2 *Suppose (M, P^M, σ_M) is a model of $T_{P,\sigma}$, a, b are tuples from M such that $a \cap \mathrm{acl}_{\mathcal{L}}(b) = \emptyset$, $a = (a_1, \ldots, a_l)$, $1 \leq i < j \leq l$ implies $a_i \neq a_j$, and $I \subseteq \{1, \ldots, l\}$. Then there is an extension $(N, P^N, \sigma_N) \models T_{P,\sigma}$ of (M, P^M, σ_M) satisfying that there is $a' = (a'_1, \ldots, a'_l) \in N \setminus M$ realizing $\mathrm{tp}_{\mathcal{L}}(a/b)$ such that $P(a'_i)$ for $i \in I$ and $\neg P(a'_i)$ for $i \notin I$.*

Choose a sequence $\langle a_i : 0 \le i < \omega \rangle$ as in Claim 2.2.1. Then there is an extension $(N, \sigma_N) \models T_\sigma$ of of (M, σ_M) such that N contains the a_i's for $0 \le i$. Let $a_k = \sigma_N^k(a_0)$ for each integer $k < 0$. Then $a_k = \sigma_N^k(a_0)$ for any $k \in \boldsymbol{Z}$. Since $a_0 \cap a_i = \emptyset$ for $i > 0$, we have $a_i \cap a_j = \emptyset$ for any $i,\ j \in \boldsymbol{Z}$ such that $i < j$. Now let $a_0 = (a_1', \ldots, a_l')$. Let $P^N = P^M \cup \{\sigma_N^k(a_i') : k \in \boldsymbol{Z},\ i \in I\}$. Then σ_N is an L_P-automorphism. We have Claim 2.2.2.

Now, suppose (M, P^M, σ_M) is a model of $T_{P,\sigma}$. With Claim 2.2.2, a standard argument shows that there is an extension $(N, P^N, \sigma_N) \models T_{P,\sigma}$ of (M, P^M, σ_M) such that $(N, P^N) \models T_P^*$ using Fact 1.3. □

Theorem 2.3 *Let T be a complete theory in a language $\mathcal{L}$. Suppose T is model complete, stable, and the model companion T_P^* of T in the language L_P exists. Then T_P^* has the PAPA.*

Proof. Let (M_0, P_0, σ_0), (M_1, P_1, σ_1), (M_2, P_2, σ_2) be models of $T_{P,\sigma}$ and suppose that (M_1, P_1, σ_1) and (M_2, P_2, σ_2) are extensions of (M_0, P_0, σ_0). We can assume that M_1 and M_2 are independent over M_0 in a big model of T. Since T is stable, $\sigma_1 \cup \sigma_2$ is an $\mathcal{L}$-elementary map on $M_1 \cup M_2$ and thus there is $(M_3 \models T$ and $\sigma_3 \in \mathrm{Aut}_{\mathcal{L}}(M_3)$ such that (M_3, σ_3) is an extension of both (M_1, P_1, σ_1) and (M_2, P_2, σ_2). Let $P_3 = P_1 \cup P_2$. Then $(M_3, P_3, \sigma_3) \models T_{P,\sigma}$. By Theorem 2.2, it embeds in a model of $(T_P^*)_\sigma$. □

Theorem 2.4 *Let T be a complete theory in a language $\mathcal{L}$. Suppose T is model complete, stable, and the model companion T_P^* of T in the language L_P exists. If T_P^* is unstable then $(T_P^*)_\sigma$ and $T_{P,\sigma}$ has no model companion.*

Proof. By Fact 1.2 and Theorem 2.3. □

In Theorem 2.3, it is sufficient to assume that T has the PAPA and any $(M, \sigma) \models T_\sigma$ is a strong amalgamation base for T_σ. In general, a subset A of a model of a theory U is a strong amalgamation base for U if $A \subset M_1, M_2$ are models of U then there is a M_3 of U and an embedding $h : M_2 \to M_3$ such that $M_1 \subset M_3$, h fixes A pointwise, and $M_1 \cap h(M_2) = A$. Also, we can conclude that T_P^* has the PAPA and $(M, P, \sigma) \models T_{P,\sigma}$ is a strong amalgamation base for $T_{P,\sigma}$. Therefore, we can repeatedly use Theorem 2.3 to show that a theory with several generic predicates (the model companion of a theory with several new predicates) has the PAPA.

3 Two Commuting Automorphisms

Let T be a complete theory in a language $\mathcal{L}$ and σ, τ new unary function symbols. Let $\mathcal{L}_\sigma = \mathcal{L} \cup \{\sigma\}$ and $\mathcal{L}_{\sigma,\tau} = \mathcal{L} \cup \{\sigma, \tau\}$. Suppose the model companion T_σ^* of $T \cup \{$"σ is an $\mathcal{L}$-automorphism"$\}$ exists. If T is stable and admits quantifier elimination, Chatzidakis and Pillay showed that T_σ^* is simple if it exists. They gave a mild assumption under which T_σ^* will be unstable. We tried to show that there is no model companion for $(T_\sigma^*)_\tau \cup \{$"$\tau$ is an $\mathcal{L}_\sigma$-automorphism"$\}$. Note that τ is an $\mathcal{L}_\sigma$-automorphism if and only if τ and σ are two commuting $\mathcal{L}$-automorphisms. Although we have not succeed to show it, we present some argument towards the proof. Main obstacle is that it is not clear if we can expand two commuting automorphisms to commuting automorphisms over some algebraic extensions.

First of all, we give a proof for the fact that there is no model companion for the theory of fields with two commuting automorphisms based on [1]. Note that the theory of fields is essentially the universal part of the theory of algebraically closed fields, which is stable.

Lemma 3.1 *Let T be the theory of fields with two commuting automorphisms. If (F, σ, τ) is an existentially closed model of T then for any integer $n \geq 2$ there is c in F such that $\sigma(c) = \tau(c)$, $c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^{n-1}(c) = 0$, and $c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^{k-1}(c) \neq 0$ for any $k < n$.*

Proof. Let t_0, t_1, ..., t_{n-2} be transcendental and algebraically independent over F. Let $t_{n-1} = -(t_0 + t_1 + \cdots + t_{n-2})$ Then t_1, ..., t_{n-2}, t_{n-1} are also transcendental and algebraically independent over F. Hence we can expand σ and τ so that $\sigma(t_i) = \tau(t_i) = t_{i+1}$ for $i = 0, 1, \ldots, n-2$. Then we have $\sigma(t_0) = \tau(t_0)$ and $t_0 + \sigma(t_0) + \cdots + \sigma^{n-1}(t_0) = 0$. σ and τ commute on $F(t_0, t_1, \ldots, t_{n-2})$. Since (E, σ, τ) is an existentially closed model of T, we can pull down t_0 in F to find c satisfying the conclusion of the lemma. □

Theorem 3.2 (Hrushovski) *There is no model companion of the theory of fields with two commuting automorphisms.*

Proof. Let ζ be a primitive third root of unity and suppose that ζ does not belong to the prime field (characteristic 2 (mod 3), or 0). Let K_0 be an algebraic closure of the prime field and σ_0 be an automorphism of K_0 such that $\sigma_0(\zeta) = \zeta^2$.

Now, suppose that T^* is a model companion of the theory of fields with two commuting automorphisms. Extend $(K_0, \sigma_0, \sigma_0)$ to $(K, \sigma, \tau) \models T^*$. We can assume that (K, σ, τ) is $\aleph_1$-saturated.

Claim 3.2.1 *In (K, σ, τ),*

$$\begin{aligned} &\sigma(z) = \tau(z),\ z + \sigma(z) + \sigma^2(z) + \cdots + \sigma^k(z) \neq 0 \text{ for } k < \omega \\ &\quad \vdash \exists x \exists y [\sigma(x) = \tau(x) = x + z \wedge y^3 = x \wedge \tau(y) = \zeta\sigma(y)] \end{aligned}$$

Let $c \in K$ be such that $\sigma(c) = \tau(c)$, $c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^k(c) \neq 0$ for $k < \omega$. Note that such c exists by Lemma 3.1.

Let E be a countable subfield of K such that $c \in E$ and $(E, \sigma|E, \tau|E)$ is an elementary substructure of (K, σ, τ). Let a be a transcendental element over E. Then we can expand $\sigma|E$ and $\tau|E$ to automorphisms σ' and τ' respectively on $E(a)$ so that $\sigma'(a) = \tau'(a) = a + c$. Then $\sigma'^n(a) = \tau'^n(a) = a + c + \sigma'(c) + \cdots \sigma'^{n-1}(c)$ and

$\sigma'^i(a) \neq \sigma'^j(a)$ if $i \neq j$. Since a has no third root in $E(a)$ and $\zeta \in E$, $X^3 - \sigma^i(a)$ is irreducible over $E(a)$. For each i, let b_i be a third root of $\sigma^i(a)$. Then we can expand σ' and τ' so that

$$\begin{aligned}\sigma(b_i) &= b_{i+1}, \\ \tau(b_i) &= \zeta b_{i+1} \quad (i \text{ is even}), \\ \tau(b_i) &= \zeta^2 b_{i+1} \quad (i \text{ is odd}).\end{aligned}$$

Let E' be a field obtained by adjoining all b_i for $i \in \boldsymbol{Z}$ to $E(a)$. If i is even then $\sigma\tau(b_i) = \sigma(\zeta b_{i+1}) = \zeta^2 b_{i+2}$, $\tau\sigma(b_i) = \tau(b_{i+1}) = \zeta^2 b_{i+2}$. If i is odd then $\sigma\tau(b_i) = \sigma(\zeta^2 b_{i+1}) = \zeta b_{i+2}$, $\tau\sigma(b_i) = \tau(b_{i+1}) = \zeta b_{i+2}$. Therefore, we have $\sigma\tau = \tau\sigma$ on E'. Hence, the RHS of the claim holds in E'. Since (E, σ, τ) is existentially closed, the RHS of the claim holds in E. We have the claim.

By compactness, there is n_0 such that

$$\begin{aligned}&\sigma(z) = \tau(z),\ z + \sigma(z) + \sigma^2(z) + \cdots + \sigma^k(z) \neq 0 \text{ for } k < n_0 \\ &\quad \Rightarrow \exists x \exists y[\sigma(x) = \tau(x) = x + z \wedge y^3 = x \wedge \tau(y) = \zeta\sigma(y)]\end{aligned}$$

in (K, σ, τ). By Lemma 3.1, we can choose c such that $\sigma(c) = \tau(c)$, $c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^k(c) \neq 0$ for $k < n_0$ but $c + \sigma(c) + \sigma^2(c) + \cdots + \sigma^{n-1}(c) = 0$ for some odd number n. Let a, b be such that $\sigma(a) = \tau(a) = a + c$, $b^3 = a$, and $\tau(b) = \zeta\sigma(b)$. Then $\sigma^n(a) = \tau^n(a) = a$. Since $\sigma^n(b)$ is a third root of a, we can write $\sigma^n(b) = \zeta^i b$ for some i.

Calculate $\sigma^n\tau(b)$ in two ways:

$$\begin{aligned}\sigma^n\tau(b) &= \sigma^n(\zeta\sigma(b)) \\ &= \sigma^n(\zeta)\sigma^{n+1}(b) \\ &= \sigma^n(\zeta)\sigma\sigma^n(b) \\ &= \sigma^n(\zeta)\sigma(\zeta^i b) \\ &= \sigma^n(\zeta)\sigma(\zeta^i)\sigma(b). \\ \sigma^n\tau(b) &= \tau\sigma^n(b) \\ &= \tau(\zeta^i b) \\ &= \sigma(\zeta^i)\zeta\sigma(b).\end{aligned}$$

Therefore, $\sigma^n(\zeta) = \zeta$ and thus n must be even. This is a contradiction. □

Since the fields are essentially the substructures of algebraically closed fields and the theory of algebraically closed fields is stable, we can conjecture that if T is stable (with some additional assumption) then there is no model companion for $T_\forall \cup \{\sigma \text{ and } \tau \text{ are commuting automorphisms}\}$.

Suppose T is stable, admits quantifier elimination, and there is a model M of T and tuples a, b in a big model of T such that $a \mathop{\smile\!\!\!\!\!\!\;|}_M b$ and $\mathrm{acl}(M, a, b) \neq \mathrm{dcl}(\mathrm{acl}(M, a) \cup \mathrm{acl}(M, b))$. Chatzidakis and Pillay [2] showed that the model companion of T_σ is unstable in this case. With the same assumption, we will try to show that there is no model companion for $T_\forall \cup \{\sigma$ and τ are commuting automorphisms$\}$.

For the sake of simplicity, we assume that T is countable. We can assume that M is countable. Let $e \in \operatorname{acl}(M, a, b) \setminus \operatorname{dcl}(\operatorname{acl}(M, a) \cup \operatorname{acl}(M, b))$. Let $\varphi(x, a, b)$ be a formula isolating $\operatorname{tp}(e/Mab)$. Let $\overline{e}$ be an enumeration of all realizations of $\varphi(x, a, b)$.

Let $\{b_i : i \in \boldsymbol{Z}\}$ be a Morley sequence for $\operatorname{tp}(b/\operatorname{acl}(aM))$ and let $\overline{e_i}$ be an enumeration of all realizations of $\varphi(x, a, b_i)$ for each i in $\boldsymbol{Z}$. Then $\{b_i\overline{e_i} : i \in \boldsymbol{Z}\}$ is independent over $\operatorname{acl}(aM)$. For each i in $\boldsymbol{Z}$, let σ_i be an automorphism such that it is identity on $\operatorname{acl}(aM)b_i$, $\sigma_i(\overline{e_i}) = e_i$ if $i \geq 0$ and $\sigma_i(\overline{e_i}) \neq e_i$ if $i < 0$. Since $\operatorname{tp}(b_i\overline{e_i}/\operatorname{acl}(Ma))$ is stationary by the elimination of imaginaries, there is an automorphism σ such that σ is an extension of all σ_i for i in $\boldsymbol{Z}$. Therefore, we have a countable extension $N \supset Ma \cup \{b_i, e_i : i \in \boldsymbol{Z}\}$ and an $\mathcal{L}$-automorphism of N such that σ fixes $Ma \cup \{b_i : i \in \boldsymbol{Z}\}$ pointwise and $\sigma(e_i) = e_i$ (as tuples) if and only if $i \geq 0$.

Let τ be an $\mathcal{L}$-automorphism such that τ fixes M pointwise and $\tau(b_i) = b_{i+1}$ for $i \in \boldsymbol{Z}$. Let $N_0 = N$, and for $i > 0$, let N_i be a model of T such that N_i is independent from $M \cup \bigcup_{j<i} N_j$ over $\operatorname{acl}(M \cup \{b_i : i \in \boldsymbol{Z}\})$ and realizes $\sigma\operatorname{tp}(N_{i-1}/M \cup \bigcup_{j<i-1} N_j)$. τ can be extended to an $\mathcal{L}$-automorphism such that $\tau(N_i) = N_{i+1}$. Let $N_i = \tau^i(N)$ for $i < 0$. Then $\{N_i : i \in \boldsymbol{Z}\}$ is an independent set over $\operatorname{acl}(M \cup \{b_i : i \in \boldsymbol{Z}\})$. Extend σ to every N_i for $i \in \boldsymbol{Z}$ through τ. Then σ is an elementary map on $\bigcup_{i \in \boldsymbol{Z}} N_i$ and σ and τ commute on $\bigcup_{i \in \boldsymbol{Z}} N_i$. Let $K = \operatorname{dcl}(\bigcup_{i \in \boldsymbol{Z}} N_i)$. Then $K \models T_\forall$, and σ and τ can be extended to $\mathcal{L}$-automorphisms of K so that they are commuting.

Note that (K, σ, τ) has the order property. Let $a_i = \tau^i(a)$ for $i \in \boldsymbol{Z}$. Consider a formula $r(x, y, x', y')$ expressing that σ pointwise fixes every realization of $\varphi(z, x', y)$. Then $r(a_i, b_i, a_j, b_j)$ if and only if $i \leq j$. Note that $r(a_i, b_i, a_j, b_j)$ and $\neg r(a_j, b_j, a_i, b_i)$ if and only if $i < j$.

Now, assume that there is a model companion T^* of

$$T_\forall \cup \{\sigma \text{ and } \tau \text{ are commuting automorphisms}\}.$$

By extending, we can assume that (K, σ, τ) is a model of T^*. Also, we can assume that (K, σ, τ) is $\aleph_i$-saturated.

Let $R(x, y, x', y') \equiv (r(x, y, x', y') \wedge \neg r(x', y', x, y))$.

We want to show the following claim in (K, σ, τ):

$$\{R(a_i, b_i, u, v) : i < \omega\} \vdash \exists x, y\, [R(a_0, b_0, x, y) \wedge R(x, y, u, v) \wedge \tau(x, y) = (x, y)]$$

If we have this claim, then we get a contradiction by compactness and the fact that τ is an $\mathcal{L}_\sigma$-automorphism.

Let (x_0, y_0) realize a non-forking extension of $\operatorname{tp}_{\mathcal{L}}(a_0, b_0/M)$ to K. Since $\operatorname{tp}_{\mathcal{L}}(a_0, b_0/M)$ is stationary, $\operatorname{tp}(x_0, y_0/K)$ is fixed by τ. If we can extend σ and τ to some extension of K so that they are commuting, $R(a_i, b_i, x_0, y_0)$ for $i < \omega$ and $R(x_0, y_0, u, v)$, we are done since (K, σ, τ) is existentially closed. But, it seems very difficult to do this in an abstract situation like this.

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